DEMYSTIFYING A DIVISIBILITY PROPERTY OF THE KOSTANT PARTITION FUNCTION

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Abstract. We study a family of identities regarding a divisibility property of the Kostant partition function which first appeared in a paper of Baldoni and Vergne. To prove the identities, Baldoni and Vergne used techniques of residues and called the resulting divisibility property "mysterious." We prove these identities entirely combinatorially and provide a natural explanation of why the divisibility occurs. We also point out several ways to generalize the identities.

1. Introduction

The objective of this paper is to provide a natural combinatorial explanation of a divisibility property of the Kostant partition function. The question of evaluating Kostant partition functions has been subject of much interest, without a satisfactory combinatorial answer. To mention perhaps the most famous such case: it is known that

$$K_{A_n^+}(1,2,\ldots,n,-\binom{n+1}{2})=\prod_{k=1}^{n} C_k,$$

where $C_k=\frac{1}{k+1}\binom{2k}{k}$ is the Catalan number, yet there is no combinatorial proof of the above identity!

Given such lack of understanding of the evaluation of the Kostant partition function, it seems a worthy proposition to provide a simple explanation for its certain divisibility properties. We explore divisibility properties of Kostant partition functions of types A_n and C_{n+1}, noting that such properties in types B_{n+1} and D_{n+1} are easy consequences of the type C_{n+1} case. The type A_n family of identities we study first appeared in a paper by Baldoni and Vergne [BV], where the authors prove the identities using residues, and where they call the divisibility property "mysterious." It is our hope that the combinatorial argument we provide successfully demystifies the divisibility property of the Kostant partition function and provides a natural explanation why things happen the way they do.

The outline of the paper is as follows. In Section 2 we define Kostant partition functions of type A_n and prove the Baldoni-Vergne identities combinatorially. Our proof also yields an affirmative answer to a question of Stanley [S] regarding a possible bijective proof of a special case of the Baldoni-Vergne identities. In Section 3 we define Kostant partition functions of type C_{n+1},

relate them to flows, and show how to modify our proof for the Baldoni-Vergne identities to obtain their analogues for type C_{n+1}. We also point out other possible variations of these identities in the type C_{n+1} case.

2. The Baldoni-Vergne identities

Before stating the Baldoni-Vergne identities, we need a few definitions. Throughout this section the graphs G we consider are on the vertex set $[n+1]$ with possible multiple edges, but no loops. Denote by m_{ij} the multiplicity of edge (i,j), $i<j$, in G. To each edge (i,j), $i<j$, of G, associate the positive type A_n root e_i-e_j, where e_i is the i^{th} standard basis vector. Let $\{\{\alpha_1,\ldots,\alpha_N\}\}$ be the multiset of vectors corresponding to the multiset of edges of G as described above. Note that $N=\sum_{1\leq i<j\leq n+1} m_{ij}$.

The **Kostant partition function** K_G evaluated at the vector $\mathbf{a}\in\mathbb{Z}^{n+1}$ is defined as

$$K_G(\mathbf{a})=\#\{(b_i)_{i\in[N]}\mid \sum_{i\in[N]} b_i\alpha_i=\mathbf{a} \text{ and } b_i\in\mathbb{Z}_{\geq 0}\}. \tag{1}$$

That is, $K_G(\mathbf{a})$ is the number of ways to write the vector $\mathbf{a}$ as a nonnegative linear combination of the positive type A_n roots corresponding to the edges of G, without regard to order. Note that in order for $K_G(\mathbf{a})$ to be nonzero, the partial sums of the coordinates of $\mathbf{a}$ have to satisfy $v_1+\ldots+v_i\geq 0$, $i\in[n]$, and $v_1+\ldots+v_{n+1}=0$.

We now proceed to state and prove Theorem 1 which first appeared in [BV]. Baldoni and Vergne gave a proof of it using residues, and called the result "mysterious." We provide a natural combinatorial explanation of the result. Our explanation also answers a question of Stanley in affirmative, which he posed in [S], regarding a possible bijective proof of a special case of the Baldoni-Vergne identities.

For brevity, we write $G-e$, or $G-\{e_1,\ldots,e_k\}$, to mean a graph obtained from G with edge e, or edges $e_1,\ldots,e_k$, deleted.

Theorem 1. *([BV]) Given a connected graph G on the vertex set $[n+1]$ with $m_{n-1,n}=m_{n-1,n+1}=m_{n,n+1}=1$ and such that*

$$\frac{m_{j,n-1}+m_{j,n}+m_{j,n+1}}{m_{j,n-1}}=c,$$

for some constant c independent of j for $j\in[n-2]$, we have that for any $\mathbf{a}=(a_1,\ldots,a_n,-\sum_{i=1}^n a_i)\in\mathbb{Z}^{n+1}$

$$K_G(\mathbf{a})=(\frac{a_1+\cdots+a_{n-2}}{c}+a_{n-1}+1)K_{G-(n-1,n)}(\mathbf{a}). \tag{2}$$

Before proceeding to the formal proof of Theorem 1 we outline it, to fully expose the underlying combinatorics. Rephrasing equation (1), $K_G(\mathbf{a})$

counts the number of **flows** $\mathbf{f}_G = (b_i)_{i\in N}$ on graph G satisfying

$$\sum_{i\in[N]} b_i\alpha_i = \mathbf{a} \text{ and } b_i \in \mathbb{Z}_{\geq 0}.$$

In the proof of Theorem 1 we introduce the concept of **partial flows** $\mathbf{f}_H$ and the following are the key statements we prove:

- The elements of the set of partial flows are in bijection with the flows on $G-(n-1,n)$ that the Kostant partition function $K_{G-(n-1,n)}(\mathbf{a})$ counts. That is,
$$\# \text{ partial flows} = K_{G-(n-1,n)}(\mathbf{a}).$$
- The elements of the multiset of partial flows $\mathbf{f}_H$, where the cardinality of the multiset is $\frac{a_1+\cdots+a_{n-2}}{c}+a_{n-1}+1$ times the cardinality of the set of partial flows, are in bijection with the flows on G that the Kostant partition function $K_G(\mathbf{a})$ counts. That is,
$$(\frac{a_1+\cdots+a_{n-2}}{c}+a_{n-1}+1)\# \text{ partial flows} = K_G(\mathbf{a}).$$

The above two statements imply a bijection between the elements of the multiset of flows on $G-(n-1,n)$ that the Kostant partition function $K_{G-(n-1,n)}(\mathbf{a})$ counts, where the cardinality of the multiset is $\frac{a_1+\cdots+a_{n-2}}{c}+a_{n-1}+1$ times the cardinality of the set of flows counted by $K_{G-(n-1,n)}(\mathbf{a})$, and the flows on G that the Kostant partition function $K_G(\mathbf{a})$ counts, yielding

$$K_G(\mathbf{a}) = (\frac{a_1+\cdots+a_{n-2}}{c}+a_{n-1}+1)K_{G-(n-1,n)}(\mathbf{a}).$$

We now proceed to the formal proof of Theorem 1.

Proof of Theorem 1. Let $\{\{\alpha_1,\ldots,\alpha_N\}\}$ be the multiset of vectors corresponding to the edges of G. Let $\alpha_N = e_{n-1}-e_n, \alpha_{N-1} = e_{n-1}-e_{n+1}$, and $\alpha_{N-2} = e_n - e_{n+1}$. Then equation (2) can be rewritten as

$$\begin{aligned} &\#\{(b_i)_{i\in[N]} \mid \sum_{i=1}^{N} b_i\alpha_i = \mathbf{a}\} = \\ &(\frac{a_1+\cdots+a_{n-2}}{c}+a_{n-1}+1)\#\{(b_i)_{i\in[N-1]} \mid \sum_{i=1}^{N-1} b_i\alpha_i = \mathbf{a}\}. \end{aligned} \tag{3}$$

Consider a flow $\mathbf{f}_H = (b_i)_{i\in[N-3]}$, $b_i \in \mathbb{Z}_{\geq 0}$, of the edges of the graph $H := G-\{(n-1,n),(n-1,n+1),(n,n+1)\}$. We call $\mathbf{f}_H$ **partial** if

$$\sum_{i=1}^{N-3} b_i\alpha_i = (a_1,\ldots,a_{n-2},x_1,x_2,x_3),$$

for some $x_1, x_2, x_3 \in \mathbb{Z}$.

Notice that given a partial flow $\mathbf{f}_H = (b_i)_{i\in[N-3]}$, it can be extended uniquely to a flow $\mathbf{f}_{G-\{(n-1,n)\}} = (b_i)_{i\in[N-1]}$, $b_i \in \mathbb{Z}_{\geq 0}$, on $G - \{(n-1,n)\}$ such that $\sum_{i=1}^{N-1} b_i\alpha_i = \mathbf{a}$. Furthermore, this correspondence is a bijection. Therefore,

$$\#\{(b_i)_{i\in[N-1]} \mid \sum_{i=1}^{N-1} b_i\alpha_i = \mathbf{a}\} = \sum_{\mathbf{f}_H} 1, \tag{4}$$

where the summation runs over all partial flows $\mathbf{f}_H$.

Also, given a partial flow $\mathbf{f}_H$ with $Y_i(\mathbf{f}_H)$, $i \in \{n-1, n, n+1\}$, denoting the total **inflow** into vertex $i \in \{n-1, n, n+1\}$ in H, that is the sum of all the flows b_i on edges of H incident to $i \in \{n-1, n, n+1\}$, the partial flow $\mathbf{f}_H$ can be extended in $Y_{n-1}(\mathbf{f}_H) + a_{n-1} + 1$ ways to a flow $\mathbf{f}_G = (b_i)_{i\in[N]}$, $b_i \in \mathbb{Z}_{\geq 0}$, of G such that $\sum_{i=1}^{N} b_i\alpha_i = \mathbf{a}$. Furthermore, given a flow $\mathbf{f}_G = (b_i)_{i\in[N]}$, $b_i \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{N} b_i\alpha_i = \mathbf{a}$, there is a unique partial flow $\mathbf{f}_H = (b_i)_{i\in[N-3]}$ from which it can be obtained. Therefore,

$$\#\{(b_i)_{i\in[N]} \mid \sum_{i=1}^{N} b_i\alpha_i = \mathbf{a}\} = \sum_{\mathbf{f}_H}(Y_{n-1}(\mathbf{f}_H) + a_{n-1} + 1), \tag{5}$$

where the summation runs over all partial flows $\mathbf{f}_H$.

Note that since

$$\frac{m_{j,n-1} + m_{j,r} + m_{j,n+1}}{m_{j,n-1}} = c,$$

for some constant c independent of j for $j \in [n-2]$, it follows that

$$c\sum_{\mathbf{f}_H} Y_{n-1}(\mathbf{f}_H) = \sum_{\mathbf{f}_H}(Y_{n-1}(\mathbf{f}_H) + Y_n(\mathbf{f}_H) + Y_{n+1}(\mathbf{f}_H)) = \sum_{\mathbf{f}_H}(a_1 + \cdots + a_{n-2}),$$

that is

$$\sum_{\mathbf{f}_H} Y_{n-1}(\mathbf{f}_H) = \sum_{\mathbf{f}_H} \frac{(a_1 + \cdots + a_{n-2})}{c}. \tag{6}$$

Thus, equation (5) can be rewritten as

$$\#\{(b_i)_{i\in[N]} \mid \sum_{i=1}^{N} b_i\alpha_i = \mathbf{a}\} = \sum_{\mathbf{f}_H} (\frac{(a_1+\cdots+a_{n-2})}{c} + a_{n-1}+1) \tag{7}$$

$$= (\frac{(a_1+\cdots+a_{n-2})}{c} + a_{n-1}+1) \sum_{\mathbf{f}_H} 1 \tag{8}$$

$$= (\frac{(a_1+\cdots+a_{n-2})}{c} + a_{n-1}+1) \#\{(b_i)_{i\in[N-1]} \mid \sum_{i=1}^{N-1} b_i\alpha_i = \mathbf{a}\}, \tag{9}$$

where the first equality uses equations (5) and (6), and the third equality uses equation (4). □

3. Type C_{n+1} Kostant partition functions and the Baldoni-Vergne identities

We now show two generalizations of Theorem 1 in the type C_{n+1} case. We first give the necessary definitions and explain the notion of flow in the context of signed graphs. Throughout this section the graphs G on the vertex set $[n+1]$ we consider are signed, that is there is a sign $\epsilon \in \{+,-\}$ assigned to each of its edges, with possible multiple edges, and all loops labeled positive. Denote by $(i,j,-)$ and $(i,j,+)$, $i \leq j$, a negative and a positive edge, respectively. Denote by m_{ij}^{ϵ} the multiplicity of edge (i,j,ϵ) in G, $i \leq j$, $\epsilon \in \{+,-\}$. To each edge (i,j,ϵ), $i \leq j$, of G, associate the positive type C_{n+1} root $\mathrm{v}(i,j,\epsilon)$, where $\mathrm{v}(i,j,-) = e_i - e_j$ and $\mathrm{v}(i,j,+) = e_i + e_j$. Let $\{\{\alpha_1,\ldots,\alpha_N\}\}$ be the multiset of vectors corresponding to the multiset of edges of G as described above. Note that $N = \sum_{1\leq i<j\leq n+1}(m_{ij}^{-} + m_{ij}^{+})$.

The **Kostant partition function** K_G evaluated at the vector $\mathbf{a} \in \mathbb{Z}^{n+1}$ is defined as

$$K_G(\mathbf{a}) = \#\{(b_i)_{i\in[N]} \mid \sum_{i\in[N]} b_i\alpha_i = \mathbf{a} \text{ and } b_i \in \mathbb{Z}_{\geq 0}\}.$$

That is, $K_G(\mathbf{a})$ is the number of ways to write the vector $\mathbf{a}$ as a nonnegative linear combination of the positive type C_{n+1} roots corresponding to the edges of G, without regard to order.

Just like in the type A_n case, we would like to think of the vector $(b_i)_{i\in[N]}$ as a **flow**. For this we here give a precise definition of flows in the type C_{n+1} case, of which type A_n is of course a special case.

Let G be a signed graph on the vertex set $[n+1]$. Let $\{\{e_1,\ldots,e_N\}\}$ be the multiset of edges of G, and $\{\{\alpha_1,\ldots,\alpha_N\}\}$ the multiset of vectors corresponding to the multiset of edges of G. Fix an integer vector $\mathbf{a} = (a_1,\ldots,a_n,a_{n+1}) \in \mathbb{Z}^{n+1}$. A **nonnegative integer a-flow $\mathbf{f}_G$** on G is a vector $\mathbf{f}_G = (b_i)_{i\in[N]}$, $b_i \in \mathbb{Z}_{\geq 0}$ such that for all $1 \leq i \leq n+1$, we have

$$\sum_{e\in E, \mathrm{inc}(e,v)=-} b(e) + a_v = \sum_{e\in E, \mathrm{inc}(e,v)=+} b(e) + \sum_{e=(v,v,+)} b(e), \tag{10}$$

where $b(e_i) = b_i$, $\mathrm{inc}(e,v) = -$ if edge $e = (g, v, -)$, $g < v$, and $\mathrm{inc}(e, v) = +$ if $e = (g, v, +)$, $g < v$, or $e = (v, j, \epsilon)$, $v < j$, and $\epsilon \in \{+, -\}$.

Call $b(e)$ the **flow** assigned to edge e of G. If the edge e is negative, one can think of $b(e)$ units of fluid flowing on e from its smaller to its bigger vertex. If the edge e is positive, then one can think of $b(e)$ units of fluid flowing away both from e's smaller and bigger vertex to infinity. Edge e is then a "leak" taking away $2b(e)$ units of fluid.

From the above explanation it is clear that if we are given an $\mathbf{a}$-flow $\mathbf{f}_G$ such that $\sum_{e=(i,j,+), i\leq j} b(e) = y$, then $\mathbf{a} = (a_1, \ldots, a_n, 2y - \sum_{i=1}^n a_i)$. It is then a matter of checking the defintions to see that for a signed graph G on the vertex set $[n + 1]$ and vector $\mathbf{a} = (a_1, \ldots, a_n, 2y - \sum_{i=1}^n a_i) \in \mathbb{Z}^{n+1}$, the number of nonnegative integer $\mathbf{a}$-flows on G is equal to $K_G(\mathbf{a})$.

Thinking of $K_G(\mathbf{a})$ as the number of nonnegative integer $\mathbf{a}$-flows on G, there is a straightforward generalization of Theorem 1 in the type C_{n+1} case:

Theorem 2. *Given a connected signed graph G on the vertex set $[n + 1]$ with $m^-_{n-1,n} = m^-_{n-1,n+1} = m^-_{n,n+1} = 1$, $m^+_{j,n-1} = m^+_{j,n} = m^+_{j,n+1} = 0$, for $j \in [n + 1]$, and such that*

$$\frac{m^-_{j,n-1} + m^-_{j,n} + m^-_{j,n+1}}{m^-_{j,n-1}} = c,$$

for some constant c independent of j for $j \in [n - 2]$, we have that for any $\mathbf{a} = (a_1, \ldots, a_n, 2y - \sum_{i=1}^n a_i) \in \mathbb{Z}^{n+1}$,

$$K_G(\mathbf{a}) = (\frac{a_1 + \cdots + a_{n-2} - 2y}{c} + a_{n-1} + 1) K_{G-(n-1,n)}(\mathbf{a}). \tag{11}$$

The proof of Theorem 2 proceeds analogously to that of Theorem 1. Namely, define **partial flows** $\mathbf{f}_H = (b_i)_{i\in[N-3]}$ on $H := G - \{(n-1, n, -), (n-1, n+1, -), (n, n+1, -)\}$ such that

$$\sum_{i=1}^{N-3} b_i \alpha_i = (a_1, \ldots, a_{n-2}, x_1, x_2, x_3),$$

for some $x_1, x_2, x_3 \in \mathbb{Z}$ and the sum of flows on positive edges is y.

Then, one can prove:

- The elements of the set partial flows are in bijection with the nonnegative integer $\mathbf{a}$-flows on $G - (n - 1, n)$. That is,

$$\#\text{ partial flows} = K_{G-(n-1,n)}(\mathbf{a}).$$

- The elements of the multiset of partial flows $\mathbf{f}_H$, where the cardinality of the multiset is $\frac{a_1+\cdots+a_{n-2}-2y}{c} + a_{n-1} + 1$ times the cardinality of the set of partial flows, are in bijection with the nonnegative integer $\mathbf{a}$-flows on G . That is,

$$(\frac{a_1 + \cdots + a_{n-2} - 2y}{c} + a_{n-1} + 1)\# \text{ partial flows} = K_G(\mathbf{a}).$$

The above two statements imply a bijection between the elements of the multiset of nonnegative integer $\mathbf{a}$-flows on $G-(n-1, n)$, where the cardinality of the multiset is $\frac{a_1+\cdots+a_{n-2}-2y}{c} + a_{n-1} + 1$ times the cardinality of the set of nonnegative integer $\mathbf{a}$-flows on $G - (n-1, n)$, and the nonnegative integer $\mathbf{a}$-flows on G, yielding

$$K_G(\mathbf{a}) = (\frac{a_1 + \cdots + a_{n-2} - 2y}{c} + a_{n-1} + 1)K_{G-(n-1,n)}(\mathbf{a}).$$

Note that the requirement that only negative edges are incident to the vertices $n-1, n, n+1$ in G stems from the fact that we need to make sure, in order for our counting arguments from the proof of Theorem 1 to work, that we can always assign nonnegative flows to the edges $(n-1, n, -), (n-1, n+1, -), (n, n+1, -)$ and also, that in case we are extending a partial flow $\mathbf{f}_H$ to a flow on G, we can extend it in $Y_{n-1}(\mathbf{f}_H) + a_{n-1} + 1$ ways. These properties will be satisfied, if we insure that "inflows" at the vertices $n-1$ and n are at least $-a_{n-1}$ and $-a_n$, respectively. To simplify the formulation, we will assume that there are no loops at the vertices $n-1, n, n+1$, though the following theorem could also be adopted to a somewhat more general setting.

Theorem 3. *Given a connected signed graph G on the vertex set $[n+1]$ with $m^-_{n-1,n} = m^-_{n-1,n+1} = m^-_{n,n+1} = 1$, $m^+_{i,j} = 0$, for $i, j \in \{n-1, n, n+1\}$, and such that*

$$\frac{m^\epsilon_{j,n-1} + m^\epsilon_{j,r} + m^\epsilon_{j,n+1}}{m^\epsilon_{j,n-1}} = c,$$

for $\epsilon \in \{+, -\}$ and for some constant c independent of j for $j \in [n-2]$, we have that for any $\mathbf{a} = (a_1, \ldots, a_n, 2y - \sum_{i=1}^n a_i) \in \mathbb{Z}^{n+1}$, $y \leq a_{n-1}+1, a_n+1$,

$$K_G(\mathbf{a}) = (\frac{a_1 + \cdots + a_{n-2} - 2y}{c} + a_{n-1} + 1)K_{G-(n-1,n)}(\mathbf{a}). \quad (12)$$

The proof technique of Theorem 3 is analogous to that of Theorem 1. We invite the reader to check each step of the proof of Theorem 1 and see how they can be adapted to prove Theorem 3.

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